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# Exact Bounds for Orthogonal Polynomials Associated with Exponential Weights*

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Estimates for orthogonal polynomials associated with $\exp(-x^m)$, $x$ real, $m$ even, are dealt with. © 1985 Academic Press, Inc.

Let $w(x) = \exp(-x^m)$, $x \in \mathbb{R}$, where $m$ is a fixed positive even integer and let $\{p_n\}_{n=0}^{\infty}$ denote the corresponding system of orthogonal polynomials. I conjectured in [13] that there exist constants $c_1$ and $c_2$ such that

$$w(x)\, p_n^2(x) \leqslant c_1\, n^{-1/m}, \qquad |x| \leqslant c_2 n^{1/m}, \tag{1}$$

for $n = 1, 2,\ldots$, and this was proved by my former student Bonan [1]. In this note I show how to apply recent results on orthogonal polynomials by Dombrowski [2], Dombrowski and Fricke [3], Freud [4, 5], Lew and Quarles [8], Magnus [9], Máté and Nevai [10] and Máté, Nevai and Zaslavsky [11] to prove the following improvement of (1).

THEOREM 1. *For every given $0 < c < 1$ there exists a constant $c_3 = c_3(c)$ such that*

$$w(x)\, p_n^2(x) \leqslant c_3 n^{-1/m}, \qquad |x| \leqslant c\left[\sqrt{\pi}\,\frac{\Gamma(m/2)}{\Gamma((m+1)/2)}\right]^{1/m} n^{1/m}, \tag{2}$$

*for $n = 1, 2,\ldots$.*

For $m = 2$, that is for the Hermite polynomials, it is well known that (2) is not valid with $c = 1$ anymore [19, p. 201] and thus one may reasonably expect that (2) fails for all $m$ with $c = 1$ though I cannot prove this at the present time. However, one should be able to prove that (2) fails with $c = 1$

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by exploiting Mhaskar and Saff's ideas in [12]. For $m=4$ and $m=6$, Theorem 1 was proved by Nevai [14] and Sheen [18] respectively, and it was used to obtain Plancherel–Rotach-type asymptotics for the polynomials $p_n$ in [15] and [18]. I anticipate similar applications of (2) for $m \geqslant 8$. Rahmanov [17] proved that the largest zero $X_n$ of $p_n$ satisfies

$$\lim_{n \to \infty} X_n n^{-1/m} = \left[ \sqrt{\pi} \frac{\Gamma(m/2))}{\Gamma((m+1)/2)} \right]^{1/m}$$

(conjectured by Freud [7]) and thus (2) yields an estimate for $p_n$ when $|x| \leqslant cX_n$, $0<c<1$.

Let $x_{kn}$, $k=1, 2,..., n$, denote the zeros of $p_n$.

THEOREM 2. *There exist positive constants $c_4$, $c_5$, and $c_6$ such that*

$$w(x_{kn})\, p_{n-1}^2(x_{kn}) \leqslant c_4 n^{-1/m}, \qquad k=1, 2,..., n \tag{3}$$

*and*

$$w(x_{kn})\, p_{n-1}^2(x_{kn}) \geqslant c_5\, n^{-1/m}, \qquad |x_{kn}| \leqslant c_6 n^{1/m}, \tag{4}$$

*are satisfied for $n=1, 2,...$.*

Inequality (4) was also proved by Bonan [1] who used different arguments. It is likely that $c_6$ in (4) can be replaced by

$$c\left[ \sqrt{\pi} \frac{\Gamma(m/2))}{\Gamma((m+1)/2)} \right]^{1/m}$$

where $0<c<1$ but proving this is beyond my reach at this time.

The polynomials $p_n$ satisfy the recursion formula

$$xp_n = a_{n+1} p_{n+1} + a_n p_{n-1}, \tag{5}$$

$n=0, 1,..., a_0=0$. The heart of the proof of both Theorems 1 and 2 is the magic formula

$$\sum_{k=0}^{n-1} [a_{k+1}^2 - a_k^2]\, p_k^2 = a_n^2 \left( p_{n-1} - \frac{x}{2a_n} p_n \right)^2 + a_n^2 \left( 1 - \frac{x^2}{4a_n^2} \right) p_n^2 \tag{6}$$

coupled with the asymptotics

$$a_n n^{-1/m} = \frac{1}{2} \left[ \sqrt{\pi} \frac{\Gamma(m/2)}{\Gamma((m+1)/2)} \right]^{1/m} + O(n^{-2}), \qquad n=1, 2,.... \tag{7}$$

Formula (6) is valid for every system of orthogonal polynomials generated by a recurrence of the form (5) and it was discovered by Dombrowski and Fricke [3] and generalized in Dombrowski [2]. For $m=2$, (7) is obvious, for $m=4$ it was proved by Lew and Quarles [8], $m=6$ is done in Máté and Nevai [10], whereas the general case when $m$ is an arbitrary fixed positive even integer was treated in Máté, Nevai, and Zaslavsky [11] where a theorem of A. Magnus [9] solving a conjecture of Freud [6] is applied to proving (7). The remaining ingredient of the proof of (2)–(4) comes from Freud [4, 5] who proved

$$w(x) \sum_{k=0}^{n-1} p_k^2(x) \leqslant c_7 n^{1-1/m}, \qquad x \in \mathbb{R} \tag{8}$$

and

$$w(x) \sum_{k=0}^{n-1} p_k^2(x) \geqslant c_8\, n^{1-1/m}, \qquad |x| \leqslant c_9 n^{1/m}, \tag{9}$$

$n=1, 2,...,$ with suitably choosen positive constants $c_7$, $c_8$, and $c_9$.

I can hardly resist the temptation to say "Theorems 1 and 2 easily follow from (5)–(9)." I do so and give a few hints as to the nature of the proofs. By (7), we can find two positive constants $c_{10}$ and $c_{11}$ such that

$$a_{k+1}^2 - a_k^2 \geqslant c_{10} k^{2/m-1}, \qquad k \geqslant c_{11}, \tag{10}$$

and then (4) follows from (6), (7), (9), and (10) by straightforward estimates with $c_6 = c_9$. Applying again (7) we find another constant $c_{11}$ such that

$$|a_{k+1}^2 - a_k^2| \leqslant c_{11} k^{2/m-1}, \qquad k=1, 2,.... \tag{11}$$

If $2^{N-1} \leqslant n-1 < 2^N$ then by (6) and (11)

$$\begin{aligned} a_n^2\left(1-\frac{x^2}{4a_n^2}\right) p_n^2 &\leqslant a_1^2 p_0^2 + c_{11} \sum_{k=1}^{2^N} k^{2/m-1} p_k^2 \\ &= a_1^2 p_0^2 + c_{11} \sum_{l=0}^{N-1} \sum_{k=2^l}^{2^{l+1}} k^{2/m-1} p_k^2 \\ &\leqslant a_1^2 p_0^2 + c_{11} \sum_{l=0}^{N-1} 2^{l(2/m-1)} \sum_{k=0}^{2^{l+1}} p_k^2 \end{aligned}$$

and now (2) follows from (7) and (8). The proof of (3) is essentially identical to the one of (2).

For further orientation I recommend my survey [16].

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